Borel Graphings of Analytic Equivalence Relations

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Key Definitions

Throughout, we will work on Polish spaces.

1. A **graphing** of an equivalence relation E is a (simple, undirected) graph G whose connectedness equivalence relation is equal to E, i.e.

   $$xEy \iff \text{there is a path from } x \text{ to } y \text{ in } G.$$

   If G is Borel, then we say G is a **Borel graphing** of E.

2. An equivalence relation E is **Borel graphable** if E has a Borel graphing.

3. For any pointclass Ψ (e.g., lightface Δ^1_1) we can similarly define Ψ-graphing, Ψ-graphable.

Papers

- A., The effectively theory of graphs, equivalence relations, and Polish spaces, Ph.D. thesis, 2019
- A., A.S. Kechris, P. Lutz, Borel graphable equivalence relations, Advances in Mathematics 487 (March): 110765, 2026
- A., Graphings of arithmetical equivalence relations, arXiv:2505.14920, 2025
- P. Lutz, A Borel graphable equivalence relation with no Borel graphing of diameter two, arXiv:2601.04417, 2026

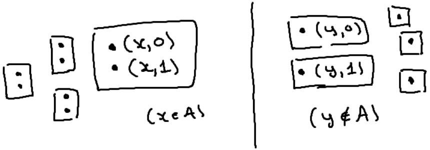
Borel graphable equivalence relations

- Of course, every Borel graph has an analytic connectedness equivalence relation.
- **Main question:** which analytic equivalence relations are Borel graphable?
  I.e., which analytic equivalence relations are the connectedness equivalence relation of some Borel graph?
- The question of Borel graphability is really only interesting for (locally) uncountable analytic equivalence relations:
  - Every Borel equivalence relation E is trivially Borel graphable. Just take $G := E - \{(x, x) : x \in X\}$).
  - A (locally) countable analytic equivalence relation E is Borel graphable if and only if E is Borel. Proof: use Lusin-Novikov.

A non-Borel graphable analytic equivalence relation

Fix some $A \subseteq X$ which is analytic, non-Borel.
Now, define E on $X \times \{0,1\}$ by

$$(x,i)E(y,j) \iff (x,i) = (y,j) \;\vee\; x = y \in A$$



If G is a graphing of E, then $\quad x \in A \iff (x,0)G(x,1)$

- But this example is locally countable.

A non-locally countable example

- For a real $x \in 2^{\mathbb{N}}$, the **Church-Kleene ordinal** of x is

$$\omega_1^x := \text{ the least (countable) ordinal which is not isomorphic to an } x\text{-computable well-ordering on } \mathbb{N}$$

- The equivalence relation E_{ck} on $2^{\mathbb{N}}$ defined by

$$x \ E_{ck} \ y \iff \omega_1^x = \omega_1^y$$

is an analytic equivalence relation with uncountable classes.

A non-locally countable example, continued

Theorem (A.)

1. *E_{ck} is not (lightface) Δ^1_1-graphable.*
2. *The equivalence relation E^{rel}_{ck} on $(2^{\mathbb{N}})^2$ defined by*

$$(x, a) E^{\mathrm{rel}}_{ck} (y, b) \iff x = y \ \& \ \omega_1^{x \oplus a} = \omega_1^{x \oplus b}$$

is not Borel graphable.

Main tool of the proof: Friedman's Conjecture (proved independently by Martin and Friedman), which in particular implies that there are no uncountable Δ^1_1 subsets of $\{x \in 2^{\mathbb{N}} : \omega_1^x = \omega_1^{\emptyset}\}$.

Borel graphability can be independent of ZFC

What about E_{ck} itself? Is it Borel graphable?

Theorem (A.-Kechris-Lutz)

E_{ck} is Borel graphable $\iff$ there is a non-constructible real.

- ($\Rightarrow$) Assuming every real is constructible, any Borel graphing is $\Delta^1_1(y)$ for some y which is hyperarithmetically least in its E_{ck}-class. Working in this class, we can replicate argument that showed E_{ck} is not Δ^1_1-graphable.
- ($\Leftarrow$) Uses Kumabe-Slaman forcing to construct a witness z to $\omega_1^x = \omega_1^y$ which is also in the same E_{ck}-class as x, y. The construction works along a branch of a tree which is a non-constructible real.

Methods to establish Borel graphability

Let E be an analytic equivalence relation on X. So,

$$xEy \iff (\exists a \in 2^{\mathbb{N}}) \underbrace{[a \text{ witnesses } xEy]}_{\text{Borel condition}}$$

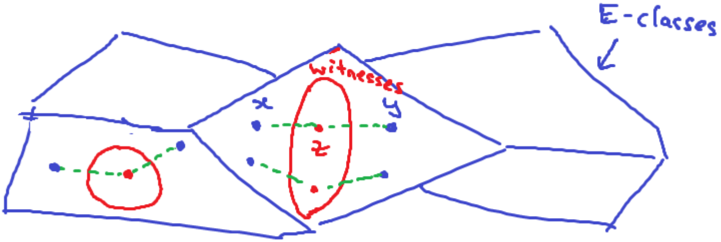
Idea: define a Borel graphing G by *restricting* the quantification $(\exists a \in 2^{\mathbb{N}})$ to make it Borel, e.g., something like (the symmetrization of)

$$xGz \iff (\exists a \leq_T z)[a \text{ witnesses } xEz]$$

where $\leq_T$ is Turing reducibility. (Could also take several jumps.)

Methods to establish Borel graphability, continued

When $xGz \iff (\exists a \leq_T z)[a \text{ witnesses } xEz]$ works, every E-class contains a special set of members which, together, have the computational power to compute all witnesses required to establish equivalence in that class.



One can think of this in the following way: the (uncountable) E-classes have a perfect set theorem, which is Borel uniformly across the classes.

An example: isomorphism of countable linear orders

Let LO be the Polish space of countable linear orderings on $\mathbb{N}$.
Let $\cong_{\mathrm{LO}}$ be the isomorphism equivalence relation on LO.

Theorem (A.-Kechris-Lutz)

$\cong_{\mathrm{LO}}$ *is Borel graphable.*

Actually, there is a more general result:

Theorem (A.-Kechris-Lutz)

For any countable first-order language $\mathcal{L}$, isomorphism of countable $\mathcal{L}$-structures is Borel graphable.

$\cong_{LO}$ is Borel graphable

Proof.

The main idea is to show that for every any two isomorphic linear orderings $\leq \cong \leq'$, there is a linear ordering $\leq''$, isomorphic to them both, such that (a code for) $\leq''$ can compute isomorphisms witnessing both $\leq \cong \leq''$ and $\leq' \cong \leq''$.

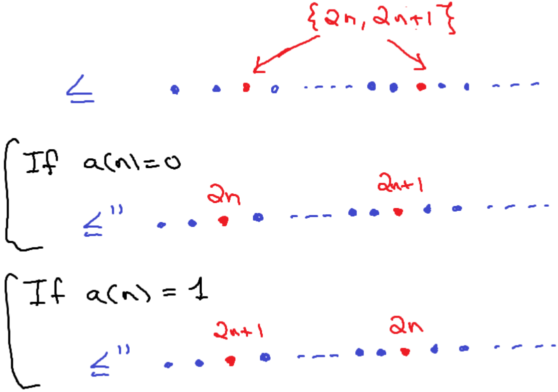
Fix a real a which codes both $\leq$ and an isomorphism $f : \leq \cong \leq'$.

It is enough that $\leq''$ is isomorphic to $\leq$ and can compute a.

This shows that a Borel graphing is (the symmetrization of)

$$\leq G \leq'' \iff (\exists f \leq_T \leq'')[f \text{ is an isomorphism between } \leq \text{ and } \leq'']$$

The construction of $\leq''$ (so that it computes a):



Graphic Polish groups

Definition

A Polish group Γ is **graphic** if every Borel action of Γ on a Polish space has a Borel graphable orbit equivalence relation.

- By a theorem of Becker and Kechris Becker, it is enough to only check the continuous actions.

Theorem (Becker-Kechris)

If $\mathcal{L}$ is a first-order language with relations of arbitrary large arities, then the logic action of S_∞ on countable $\mathcal{L}$-structures is universal for S_∞.

Theorem (A.-Kechris-Lutz)

S_∞ is a graphic Polish group.

Other graphic Polish groups

Definition

A Polish group is **tame** if all of its Polish group actions induce Borel orbit equivalence relations.

- Immediately, every tame Polish group is graphic.
- E.g., locally compact Polish groups are tame, hence graphic.

Theorem (A.-Kechris-Lutz)

Every connected Polish group is graphic.

Open: Are all Polish groups graphic?

Connected Polish groups are graphic

Proof.

(Main idea) Let Γ be a connected Polish group acting continuously on X (with no fixed points.)

For X: fix a compatible metric d, countable dense subset $\{w_i\}$.

For any $x \in X$ and any $a \in 2^{\mathbb{N}}$ (coding witness information), we will find an equivalent z such that $\bigoplus_i d(z, w_i)$ computes a.

Consider $\Gamma \to \mathbb{R}$, $g \mapsto d(g \cdot x, w_i)$. Which i?

Since we assumed no fixed points, can pick an i so that the map is non-constant.

$\implies$ Range of $g \mapsto d(g \cdot x, w_i)$ is a non-trivial interval in $\mathbb{R}$.

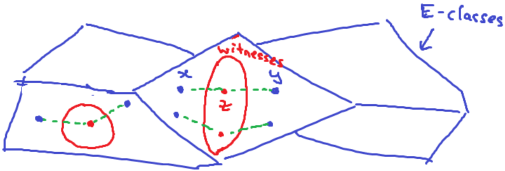
Can pick $g \in \Gamma$ so that $r := d(g \cdot x, w_i)$ computes a. $z := g \cdot x$. □

Diameters of graphings

Definition

The **diameter** of a graph is the least $k \in \mathbb{N} \cup \{\infty\}$ such that any two connected vertices have a path of length $\leq k$ between them.

In many of our results, we get Borel graphings of diameter 2:



- Not always: our proof that connected Polish groups are graphic only guarantees Borel graphings of diameter 4. But this may not be optimal.

Coding properties

Definition

1. An equivalence relation E on X has the **Borel witness coding property** if there is a Borel function $f : X \to 2^{\mathbb{N}}$ such that for all $x \in X$ and $a \in 2^{\mathbb{N}}$, there is some yEx such that $f(x)$ computes both a and a witness to the equivalence xEy.
   E.g., E_{LO} has the Borel witness coding property.
2. An equivalence relation E on X has the **Borel coding property** if there is a Borel function $f : X \to 2^{\mathbb{N}}$ such that for all $x \in X$ and $a \in 2^{\mathbb{N}}$, there is some yEx such that $f(x)$ computes a.
   E.g., A continuous action of a connected Polish group with no fixed points has an orbit equivalence relation which has the Borel coding property.

Coding properties and Borel graphings

Theorem (A.-Kechris-Lutz)

1. *The Borel witness coding property implies the Borel coding property.*
2. *If E has the Borel witness coding property, then E is Borel graphable with diameter* 2.
3. *There are Borel graphable equivalence relations (with no countable classes) which do not have the Borel coding property.*
4. *Let Γ be a Polish group acting continuously on a Polish space X and let E be the orbit equivalence relation. If E has the Borel coding property, then it is Borel graphable (with diameter 4).*

Graphings of diameter > 2

Question: is there an analytic equivalence relation which is Borel graphable but **not** Borel graphable with diameter 2?

Not Borel graphable with diameter k?

Not Borel graphable with finite diameter?

Lutz answered the first question affirmitively:

Theorem (Lutz)

There is a Borel graphable equivalence relation that has no Borel graphings of diameter 2*.*

Lutz's example

E is an equivalence relation on $\mathrm{LO} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

$(L, r, x) E (R, s, y)$ iff $L = R$ and $r = S$ and one of the following holds:

- L is ill-founded.
- L is well-founded ($|L| = \alpha$) and x, y are both $r^{(\alpha)}$-generic (but not necessarily mutually so).
- L is well-founded ($|L| = \alpha$) and neither x nor y is $r^{(\alpha)}$-generic.

[x is z-generic if it meets or avoids all $\Sigma^0_1(z)$ sets of finite bit strings.]

Sketch of Lutz's proof

Suppose towards a contradiction that G is a Borel graphing of diameter 2.

Pick $r \in 2^{\mathbb{N}}$ and L ($|L| = \alpha$) so that G is $\Sigma^0_{1+\alpha}(r)$.

If x, z are mutually $r^{(\alpha)}$-generic, then $(L+1, r, x)$ and $(L+1, r, y)$ cannot have a G-edge between them.

But—there exists a pair x, y such that for any other z, either (x, z) or (y, z) are sufficiently mutually generic

Properties of Lutz's example

(These were all proven in A.-Kechris-Lutz)

- E is Borel graphable with diameter 4.
- Unknown whether a diameter 3 Borel graphing of E exists.
- E does **not** have the Borel coding property (although it does have some weaker coding properties).
- Unclear if this example and the methods can be extended to other diameter questions.

Thank you!